OPEN ACCESS
Saudi Journal of Engineering and Technology
Abbreviated Key Title: Saudi J Eng Technol
ISSN 2415-6272 (Print) |ISSN 2415-6264 (Online)
Scholars Middle East Publishers, Dubai, United Arab Emirates
Journal homepage: https://saudijournals.com

**Review Article**

# Parameterization of the Density Component in the Composite Wave Curve in a Fluid Flow Model in a Tube with Variable Cross-Section

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**DOI:** 10.36348/sjet.2022.v07i03.004 | **Received:** 01.02.2022 | **Accepted:** 07.03.2022 | **Published:** 12.03.2022

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## Abstract

We first investigate the general properties of the system and identify all possible wave combinations. Second, a critical investigation of the Riemann problem yields definite results for large data about the existence of Riemann solutions made of Lax shocks, rarefaction waves, and admissible stationary contacts. In some range of Riemann data, no solution exists. We can identify relatively large neighborhoods of Riemann data in which the Riemann problem admits a unique solution.

**Keywords:** Composite wave, shock wave, rarefaction wave, Riemann problem.

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## I. INTRODUCTION

The curves of composite waves in the fluid flow model in a tube of variable cross-section plays an essential role in the solution of the Riemann problem. However, the variation of the curve shape has not been demonstrated, partly due to the complexity of the waves arising in one characteristic wave group along the curves of the waves in another characteristic wave group. This gives us an incentive to research. Precisely, the model is given by

$$\begin{cases} \partial_t(a\rho)+\partial_x(a\rho u)=0, \\ \partial_t(a\rho u)+\partial_x(a(u^2+p(\rho)))=p(\rho)\partial_x a, \\ \partial_t a=0. \end{cases} \quad \ldots\ldots\ldots\ldots (1.1)$$

Where the cross-sectional area of the pipe are denoted by a = a(x) (with x ∈ R). Here, ρ stand for the density, u stand for the particle velocity of the fluid under consideration, and the pressure function p = p(ρ) is given by (for simplicity we assume that the pressure given by the ideal gas equation is isentropy)

$$p=p(\rho)=\kappa\rho^{\gamma},\ \kappa>0,\ 1<\gamma<5/3. \quad \ldots\ldots (1.2)$$

Observe that the third equation in (1.1) is a trivial equation, the function a depends only on x.

Consider the function

$$h(\rho):=\frac{\kappa\gamma}{\gamma-1}\rho^{\gamma-1}$$

The function h is defined by $h'(\rho)=p'(\rho)/\rho$

The system (1.1) is equivalent to the following three conservation laws, when the smooth solutions (x, t) → (ρ, u, a):

$$\begin{cases} \partial_t(a\rho)+\partial_x(a\rho u)=0, \\ \partial_t(u)+\partial_x(\frac{u^2}{2}+h(\rho))=0, \\ \partial_t(a)=0, \end{cases} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1.3)$$

The system (1.3) can be re-written as a nonconservative system as

$$\partial_t U + A(U)\partial_x U = 0, \quad \ldots\ldots\ldots\ldots (1.4)$$

Where

$$U = \begin{pmatrix} \rho \\ u \\ a \end{pmatrix}, \; A(U) = \begin{pmatrix} u & \rho & \rho u / a \\ h'(\rho) & u & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

The matrix A = A (U) has three real eigenvalues

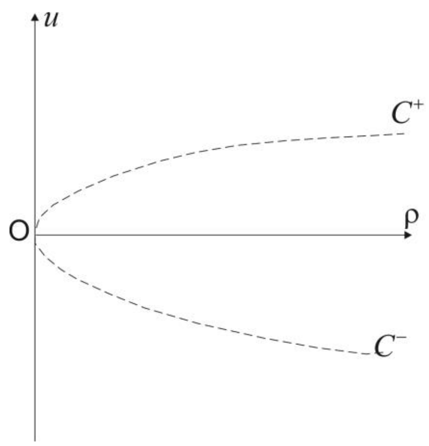
$$\begin{aligned} \lambda_1(U) &:= u - \sqrt{p'(\rho)}, \\ \lambda_2(U) &:= 0, \\ \lambda_3(U) &:= u + \sqrt{p'(\rho)}, \end{aligned} \quad \ldots\ldots\ldots\ldots (1.5)$$

Together with the corresponding eigenvectors which can be chosen as

$$r_1(U) = \begin{pmatrix} \rho \\ -\sqrt{p'(\rho)} \\ 0 \end{pmatrix}, \; r_2(U) = \begin{pmatrix} \rho u \\ -p'(\rho) \\ a(u - \frac{p'(\rho)}{u}) \end{pmatrix}, \; r_3(U) = \begin{pmatrix} \rho \\ \sqrt{p'(\rho)} \\ 0 \end{pmatrix} \ldots\ldots\ldots\ldots (1.6)$$

Thus, the system (1.1) is hyperbolic.

We call: $C^{\pm} := \left\{ (\rho, u, a) \,|\, u = \pm\sqrt{p'(\rho)} = \pm\sqrt{\kappa\gamma}\rho^{\frac{\gamma-1}{2}} \right\}$ is the sonic surface $(\rho, u)$.




Sonic surface

Moreover, the first and the third characteristic speeds can coincide, i.e.,

$$\lambda_1(U) = \lambda_2(U) = 0$$

On the surface

$$C^{+} = \left\{ (\rho, u, a) \,|\, u = \sqrt{\kappa\gamma}\rho^{\frac{\gamma-1}{2}} \right\}, \quad \ldots\ldots\ldots\ldots\ldots (1.7)$$

And the second and the third characteristic speeds can coincide, i.e.,

$$\lambda_2(U) = \lambda_3(U) = 0$$

On the surface

$$C^- = \left\{(\rho,u,a)\,|\,u = -\sqrt{\kappa\gamma}\rho^{\frac{\gamma-1}{2}}\right\}. \qquad (1.8)$$

It means that the system (1.1) is hyperbolic, but is not strictly hyperbolic.
The system (1.1) is strictly hyperbolic on regions:

$$G_1 := \left\{(\rho,u): u < -\sqrt{p'(\rho)}\right\},$$
$$G_2 = \left\{(\rho,u): |u| < \sqrt{p'(\rho)}\right\},$$
$$G_3 = \left\{(\rho,u): u > \sqrt{p'(\rho)}\right\}, \qquad (1.9)$$
$$G_2^+ = \left\{(\rho,u) \in G_2 : u > 0\right\},$$
$$G_2^- = \left\{(\rho,u) \in G_2 : u < 0\right\}.$$

In each of these regions, we have

$$\lambda_1 < \lambda_3 < \lambda_2,\ in\, G_1,$$
$$\lambda_1 < \lambda_2 < \lambda_3,\ in\, G_2,$$
$$\lambda_2 < \lambda_1 < \lambda_3, in\, G_3.$$

The 1- and the 3- characteristic families are genuinely nonlinear:

$$-\nabla\lambda_1.r_1 = \nabla\lambda_3.r_3 = \frac{1}{2\sqrt{p'(\rho)}}(\rho p''(\rho) + 2p'(\rho)) \neq 0.$$

And the second characteristic family is linearly degenerate

$$\nabla\lambda_2.r_2 = 0.$$

A shock wave (from 1- and 3-families) connecting a given left-hand state $U_0$ to a right-hand state $U$ must satisfy the Lax shock inequalities [8].

$$\lambda_i(U) < \bar{\lambda}_i(U,U_0) < \lambda_i(U_0), \quad i = 1,3 \qquad (1.10)$$

Where $\bar{\lambda}_i(U,U_0)$ is the shock speed.

Shock waves from 1- and 3-families are constrained by the jump conditions [10]. The so-called Hugoniot set is determined:

$$(u-u_0)^2 = \kappa\left(\frac{1}{\rho_0} - \frac{1}{\rho}\right)(\rho^\gamma - \rho_0^\gamma)$$

Since the 1-Hugoniot curve starts from $U_0$ and has the direction vector of the tangent at $U_0$ as $\vec{r}_1(U_0)$, we can determine the 1-Hugoniot curve:

$$\mathrm{H}_1(U_0): u = \begin{cases} u_0 - \left(\kappa\left(\frac{1}{\rho_0} - \frac{1}{\rho}\right)(\rho^\gamma - \rho_0^\gamma)\right)^{1/2}, & \rho > \rho_0, \\ u_0 + \left(\kappa\left(\frac{1}{\rho_0} - \frac{1}{\rho}\right)(\rho^\gamma - \rho_0^\gamma)\right)^{1/2}, & \rho < \rho_0. \end{cases}$$

Similar arguments lead to 3-Hugoniot curve:

$$\mathrm{H}_3(U_0): u = \begin{cases} u_0 + \left( \kappa(\frac{1}{\rho_0} - \frac{1}{\rho})(\rho^\gamma - \rho_0^\gamma) \right)^{1/2}, & \rho > \rho_0, \\ u_0 - \left( \kappa(\frac{1}{\rho_0} - \frac{1}{\rho})(\rho^\gamma - \rho_0^\gamma) \right)^{1/2}, & \rho < \rho_0. \end{cases}$$

Using Lax shock inequalities (1.10), we can define the 1-wave and a 3-wave (left-to-right) shock curve $S_1(U_o)$, $S_3(U_o)$ consisting of all right-hand states $U$ that can be connected to $U_0$ by a shock by 1-wave and a 3-wave shock:

$$\begin{aligned} S_1(U_0): \omega_1(\rho; U_0) &:= u_0 - \left( \kappa(\frac{1}{\rho_0} - \frac{1}{\rho})(\rho^\gamma - \rho_0^\gamma) \right)^{1/2}, \quad \rho > \rho_0, \\ S_3(U_0): \omega_3(\rho) &:= u_0 - \left( \kappa(\frac{1}{\rho_0} - \frac{1}{\rho})(\rho^\gamma - \rho_0^\gamma) \right)^{1/2}, \quad \rho < \rho_0. \end{aligned} \qquad \text{........... (1.11)}$$

Similarly, the 1-wave and 3-wave (right-to-left) shock curve $S_1^B(U_o)$, $S_3^B(U_o)$ consisting of all left-hand states $U$ that can be connected to $U_0$ by a Lax shock are

$$\begin{aligned} S_1^B(U_0): u &= u_0 + \left( \kappa(\frac{1}{\rho_0} - \frac{1}{\rho})(\rho^\gamma - \rho_0^\gamma) \right)^{1/2}, \quad \rho < \rho_0, \\ S_3^B(U_0): u &= u_0 + \left( \kappa(\frac{1}{\rho_0} - \frac{1}{\rho})(\rho^\gamma - \rho_0^\gamma) \right)^{1/2}, \quad \rho > \rho_0 \end{aligned} \qquad \text{.......... (1.12)}$$

Next, we search for rarefaction waves, i.e., smooth self-similar solutions to the system (1.3). As usual, we consider for ordinary differential equations

$$\begin{aligned} \frac{dU}{d\xi} &= \frac{r_i(U)}{\nabla \lambda_i . r_i(U)}, \quad i = 1, 3, \\ U(\xi_0) &= U_0. \end{aligned} \qquad \text{(1.13)}$$

...............

Therefore, we can determine the 1-wave (consisting of all right-hand states $U$ that can be connected to $U_0$) rarefaction curve $R_1(U_0)$ by

$$\begin{aligned} R_1(U_0): \omega_1(\rho) &= u_0 - \int_{\rho_0}^{\rho} \frac{\sqrt{p'(\rho)}}{\rho} d\rho \\ &= u_0 - \frac{2\sqrt{\kappa\gamma}}{\gamma - 1}(\rho^{(\gamma-1)/2} - \rho_0^{(\gamma-1)/2}), \quad \rho \le \rho_0, \quad \text{............ (1.14)} \\ \rho^{\frac{\gamma-1}{2}}(\xi) &= \rho_0^{\frac{\gamma-1}{2}} - \frac{\gamma - 1}{\sqrt{\kappa\gamma(\gamma+1)}}(\xi - \xi_0), \quad \xi \ge \xi_0. \end{aligned}$$

And the 3-wave (right-to-left) rarefaction curve $R_3(U_0)$ by

$$R_3(U_0): \omega_3(\rho) = u_0 + \int_{\rho_0}^{\rho} \frac{\sqrt{p'(\rho)}}{\rho} d\rho, \rho \le \rho_0$$
$$\rho^{\frac{\gamma-1}{2}}(\xi) = \rho_0^{\frac{\gamma-1}{2}} + \frac{\gamma-1}{\sqrt{\kappa\gamma(\gamma+1)}}(\xi - \xi_0), \quad \xi \le \xi_0 \quad \text{............... (1.15)}$$

The 1-wave (consisting of all left-hand states $U$ that can be connected to $U_0$) rarefaction curve $R_1^B(U_0)$ by

$$R_1^B(U_0): \omega_1^B(\rho) = u_0 - \int_{\rho_0}^{\rho} \frac{\sqrt{p'(\rho)}}{\rho} d\rho$$
$$= u_0 - \frac{2\sqrt{\kappa\gamma}}{\gamma-1}(\rho^{(\gamma-1)/2} - \rho_0^{(\gamma-1)/2}), \ \ \rho \ge \rho_0, \text{............ (1.16)}$$
$$\rho^{\frac{\gamma-1}{2}}(\xi) = \rho_0^{\frac{\gamma-1}{2}} - \frac{\gamma-1}{\sqrt{\kappa\gamma(\gamma+1)}}(\xi - \xi_0), \ \ \xi \le \xi_0$$

And the 3-wave (consisting of all left-hand states $U$ that can be connected to $U_0$) rarefaction curve $R_3^B(U_0)$ by

$$R_3^B(U_0): \omega_3^B(\rho) = u_0 + \int_{\rho_0}^{\rho} \frac{\sqrt{p'(\rho)}}{\rho} d\rho, \quad \rho \ge \rho_0$$
$$\rho^{\frac{\gamma-1}{2}}(\xi) = \rho_0^{\frac{\gamma-1}{2}} + \frac{\gamma-1}{\sqrt{\kappa\gamma(\gamma+1)}}(\xi - \xi_0), \quad \xi \le \xi_0 \quad \text{.............. (1.17)}$$

In conclusion, we can define the wave curves

$$\begin{aligned} W_1(U_0) &:= S_1(U_0) \cup R_1(U_0), \\ W_1^B(U_0) &:= S_1^B(U_0) \cup R_1^B(U_0), \\ W_3(U_0) &:= S_3(U_0) \cup R_3(U_0), \\ W_3^B(U_0) &:= S_3^B(U_0) \cup R_3^B(U_0). \end{aligned} \quad \text{............... (1.18)}$$

From (1.11), (1.13), (1.14), (1.15), (1.16), (1.17) and (1.18) we have equations of wave curves

$$W_1(U_0): u = \omega_1(U_0;\rho) = \begin{cases} u_0 - \dfrac{2\sqrt{\kappa\gamma}}{\gamma-1}(\rho^{(\gamma-1)/2} - \rho_0^{(\gamma-1)/2}), & \rho \le \rho_0, \\ u_0 - \left(\kappa(\dfrac{1}{\rho_0} - \dfrac{1}{\rho})(\rho^\gamma - \rho_0^\gamma)\right)^{1/2}, & \rho \ge \rho_0, \end{cases} \quad \text{............ (1.19)}$$

$$W_1^B(U_0): u = \omega_1^B(U_0;\rho) = \begin{cases} u_0 + \left(\kappa(\dfrac{1}{\rho_0} - \dfrac{1}{\rho})(\rho^\gamma - \rho_0^\gamma)\right)^{1/2}, & \rho \le \rho_0, \\ u_0 - \dfrac{2\sqrt{\kappa\gamma}}{\gamma-1}(\rho^{(\gamma-1)/2} - \rho_0^{(\gamma-1)/2}), & \rho \ge \rho_0, \end{cases} \quad \text{............ (1.19a)}$$

$$W_3(U_0): u = \omega_3(U_0;\rho) = \begin{cases} u_0 - \left(\kappa(\dfrac{1}{\rho_0} - \dfrac{1}{\rho})(\rho^\gamma - \rho_0^\gamma)\right)^{1/2}, & \rho \le \rho_0, \\ u_0 + \dfrac{2\sqrt{\kappa\gamma}}{\gamma-1}(\rho^{(\gamma-1)/2} - \rho_0^{(\gamma-1)/2}), & \rho \ge \rho_0, \end{cases} \quad \text{............... (1.19b)}$$

$$W_3^B(U_0): u = \omega_3^B(U_0;\rho) = \begin{cases} u_0 + \dfrac{2\sqrt{\kappa\gamma}}{\gamma-1}(\rho^{(\gamma-1)/2} - \rho_0^{(\gamma-1)/2}), & \rho \ge \rho_0, \\ u_0 + \left(\kappa(\dfrac{1}{\rho_0} - \dfrac{1}{\rho})(\rho^\gamma - \rho_0^\gamma)\right)^{1/2}, & \rho \le \rho_0, \end{cases} \quad \text{.......... (1.19c)}$$

Let us investigate properties of the wave curves. It follows from (1.12) that

$$\frac{d\omega_1(U_0,\rho)}{d\rho} = -\frac{\kappa\left[\dfrac{1}{\rho^2}(\rho^\gamma - \rho_0^\gamma) + (\dfrac{1}{\rho_0} - \dfrac{1}{\rho})\gamma\rho^{\gamma-1}\right]}{2|\omega_1(U_0,\rho) - u_0|} < 0, \quad \rho > \rho_0 \quad \text{................ (1.20)}$$

And, clearly,

$$\frac{d\omega_1(U_0,\rho)}{d\rho} = -\sqrt{\kappa\gamma}\,\rho^{\frac{\gamma-3}{2}} < 0, \quad \rho \le \rho_0 \quad \text{.................. (1.21)}$$

Similar calculations show that

$$\frac{d\omega_3(U_0,\rho)}{d\rho} > 0, \quad \rho > 0. \quad \text{................ (1.22)}$$

It is interesting that the shock speeds in the nonlinear characteristic fields may coincide with the characteristic speed of the linearly degenerate field as stated in the following properties.

**Lemma 1.1** *(1- and 3-wave curves) The* $W_1(U_0,a)$ *and* $W_1^B(U_0,a)$ *are strictly convex and strictly decreasing functions of a, while* $W_2(U_0,a)$ *and* $W_2^B(U_0,a)$ *are strictly concave and strictly increasing functions of a ≥ 0.*

*It is clearly that the function* $W_3(U_0,a),\ a \ge 0$ *, is strictly convex and strictly decreasing for* $u_0 > 0$*, and strictly concave and strictly increasing for* $u_0 < 0$*.*

**Lemma 1.2**

i. *If* $U_0$ *is located above the curve* $C_*$*, then the curve* $W_1(U_0)$ *has a unique intersection with each curve* $C_\pm, C_*, C^*$*. If* $U_0$ *is located below* $C_*$*, no such an intersection is available.*
ii. *If* $U_0$ *is located below or on the curve* $C^*$*, then the curve* $W_3(U_0)$ *has a unique intersection with each curve* $C_\pm, C_*, C^*$*. If* $U_0$ *is located above* $C^*$*, no such an intersection is available.*

## II. PARAMETERIZATION OF THE DENSITY COMPONENT OF THE COMPOSITE WAVE CURVE

Solution curves of the Riemann problem for (1.1) are understood to be either wave curves that were already defined in the previous section or sets of composite waves $W_{1\to2}(U_0,a),\ W_{1\to2}^B(U_0,a)$ (for 1- and 2-wave families) or $W_{2\to3}(U_0,a),\ W_{2\to3}^B(U_0,a)$ (for 2- and 3-wave families). For simplicity, we restrict our attention in this section to the situation that there is only one stationary shock in each composite wave. Two stationary shocks in a composite wave is also possible and this will be covered in another paper.

**Definition 2.1.** The composite curve $W_{1\to2}^i(U_0,a)$) is the set of all states $U = (\rho, u, a)$ such that there exists a state $U' \in W_1^i(U_0)$ and the 1-wave from $U'$ to $U$ can be followed by the stationary shock from $U'$ to $U$ by using $\varphi_i(U',a),\ i=1,2$ . The composite curve $W_{2\to1}^i(U_0,a)$ is the set of all the states $U = (\rho, u, a)$ such that the (fixed) stationary jump from $U'$ to some $U$ (using $\varphi_i(U',a),\ i=1,2$) can be followed by the 1-waves from $U'$ to U, $i=1,2$ . Under the transformation

$$\underbrace{U_0 \xrightarrow{W_1} U' \xrightarrow{W_2} U}_{W_{1\to2}^i},\ x \mapsto -x; \quad u \mapsto -u \quad \text{............... (2.1)}$$

a right-hand state $U=(\rho,u,a)$ becomes a left-hand state of the form $(\rho,-u,a)$. In generality, we can always assume from now on that

$a_0 < a$ …………….. (2.2)

We need only construct $W_{1\to2}(U_0,a)$, as similar arguments can be used for other cases. Set

$U_\pm := W_1(U_0)\cap C^\pm,\ Z(U_0)=W_1(U_0)\cap\{u=0\}$ ……………. (2.3)

***Case 1***

We assume that $U_0$ is below or on the curve $C_*$ (Figure 1). The curve $W_1(U_0)$ always remains in $G_1$ and does not cross the strict hyperbolicity boundary $C^\pm$. Therefore, all the 1-waves have negative speeds. Consequently, the 2-waves can not be followed by 1-waves. The only way is that the 1-waves are followed by 2-waves. Thus, we have in this case

$W_{1\to2}(U_0,a)\supset W^1_{1\to2}(U_0,a)\cup W^2_{1\to2}(U_0,a)$ .. (2.4)

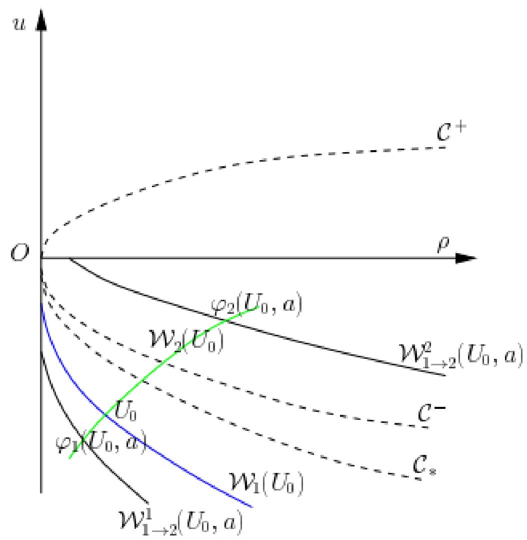



**Figure 1: $U_0$ is below or on the curve $C_*$**

***Case 2***

$U_0$ is between $C_*$ and $C^+$ or on the curve $C^+$ (Figure 2). The construction can be a 1-wave from $U_0$ to $U=(\rho,u)\in W_1(U_0)$ as long as $U$ do not belong to $G_3$, followed by a stationary jump by either using $\varphi_1(U_0,a)$ to a state $U_1$ with

$$\begin{aligned} &U_1\in G_1,\ \ if\ \ u\le 0\\ &U_1\in G_3,\ \ if\ \ u\ge 0\end{aligned}\ \ \text{……………} (2.5)$$

or using $\varphi_2(U_0,a)$ to a state $U_2\in G_2$. Such states $U_2$ form $W^2_{1\to2}(U_0,a)$. The states $U_1\in G_3$ form the set W1 1→2(U0, a). In the case $U_1\in G_3$ reached by a stationary jump from $U\in G_2^+$, Because of $U_1\in G_3$, we have

$\lambda_1(U_1)>0$ …………….. (2.6)

The construction can therefore be continued with $W_1(U_1)$ as long as the 1-shock speed from $U_1$ is non-negative. Hence, we have composite wave of two curves $W^1_{1\to2}(U_0,a),\ W^2_{1\to2}(U_0,a)$ in this case and a one-parameter family of solutions described by

$W_{1\to2}(U_0,a)\supset W^1_{1\to2}(U_0,a)\cup W^2_{1\to2}(U_0,a)$ ………… (2.7)

$U\in W_1(U_0)$, $U$ between $U_+$ and $Z(U_0)$

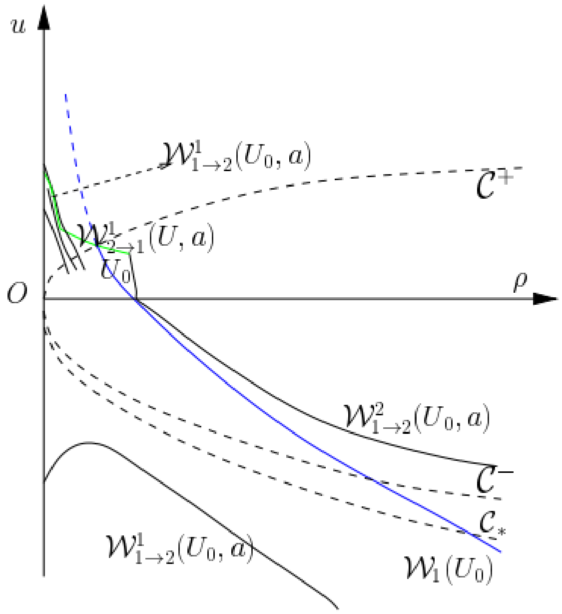



**Figure 2: $U_0$ between $C_*$ and $C^+$**

Finally, assume that $U_0$ is above $C^+$, i.e, $U_0 \in G_3$ (Figure 3).

In a neighborhood of $U_0$, the shock speed and the characteristic speed are positive, so that stationary shocks can be followed by 1-waves, only: using $\varphi_1(U_0, a)$, the solution can begin with a stationary shock to a state $U_1 \in G_3$, followed by using $W_1(U_1)$ as long as the shock speed is non-negative, i.e,

$U \in W_1(U_1) : \rho \le \psi_1(U_1)$ …………… (2.8)

This is the curve $W^1_{2\to 1}(U_0, a)$. Clearly, this curve crosses the curve $C^+$. If the solution jumps by a stationary shock using $\varphi_2(U_0, a)$ to a state $U_2$, then we know by Lemma 1.2 that $U_2 \in G_2$ in which 1-wave speeds are always negative so that 1-waves can not follow. In other words, $W^1_{2\to 1}(U_0, a)$ is empty.

On the other hand, from $U_0$, the construction can begin with a non-positive shock from $U_0$ to some state $U$ with $\rho \ge \psi_1(U_0)$, followed by a stationary shock using either $\varphi_1(U_0, a)$ to a state $U_1$ with

$$U_1 \in G_1, if\ u \le 0$$
$$U_1 \in G_3, if\ u \ge 0$$

or using $\varphi_2(U_0, a)$ to a state $U_2 \in G_2$. These two ways determine two composite curves $W^1_{1\to 2}(U_0, a)$, $W^2_{1\to 2}(U_0, a)$. When using $\varphi_1(U_0, a)$ to attain a point $U_1 \in G_3$, similarly, by virtue of (13.6) the construction can be continued by using $W_1(U_1)$ as much as 1- shock speeds from $U_1$ are non-negative. In this case, the composite wave set consists of three curves and a one-parameter family of solutons:

$W_{1,2}(U_0, a) \supset W^1_{1\to 2}(U_0, a) \cup W^2_{1\to 2}(U_0, a) \cup W^1_{2\to 1}(U_0, a) \cup W^1_{2\to 1}(U, a)$ …….. (2.9)

$U \in W_1(U_0)$, $U$ between $U_0$ and $Z(U_0)$,

Where $U_0 := (\psi_1(U_0), w_1(U_0, \psi_1(U_0)))$ the point at which the 1-shock from $U_0$ has zero speed. The construction is completed.

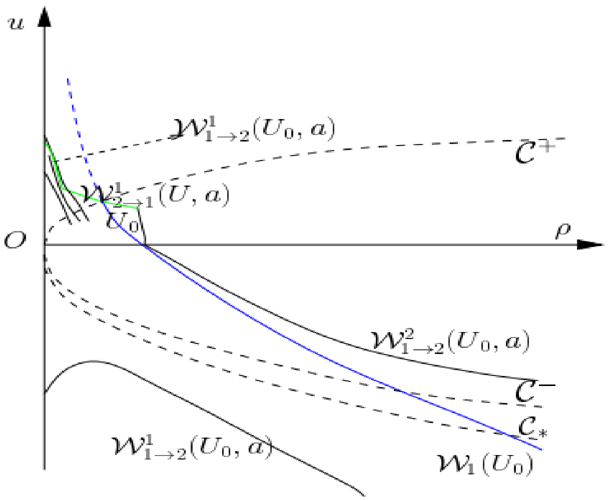



**Figure 3: $U_0$ is about $C^+$**

Obviously, the curve $W^1_{2\to 1}(U_0, a)$ is monotone, as a part of $W_1(U_1)$ for some $U_1$.

Moreover, it is derived from the above construction that the two curves $W^1_{1\to 2}(U_0, a)$, $W^2_{1\to 2}(U_0, a)$ can be parameterized by ρ:

$$\rho \mapsto \left(\varphi_i(\rho, w_1(U_0,\rho); a), w_2((\rho, w_2(U_0,\rho)); \varphi_i(\rho, w_1(U_0, a))\right), \quad i = 1, 2, \quad \text{………} \ (2.10)$$

Where $\rho < \rho^*$ is defined by the implicit equation:

$$\sum((\rho, w_1(U_0,\rho)), \varphi_i(\rho, w_1(U_0,\rho)); a) = 0$$

…………….. (2.11)

## III. CONCLUSION

Hyperbolic systems of balance laws in nonconservative form possess very interesting but rather complicated phenomena. In particular, characteristic speeds $\lambda_2$ may coincide or less than $\lambda_1$, $\lambda_3$ and the order in Riemann solutions of elementary waves such as shocks, rarefaction waves, and stationary contacts may be changed from one region to another. This can be dealt with when solving the Riemann problem by building up the curves of composite waves. In this paper we first establish the monotonicity property of these curves of composite waves and this composite waves can be parameterized by density component ρ.

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