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Deflection of Simply Supported Rectangular Plates under Shear and Bending Deformations Using Orthogonal Polynomial Function

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Abstract

Plates are common structural elements use in several engineering applications and are subjected to different types of loads, including acoustic excitations. The first satisfactory theory of bending plates is associated with Navier and later Kirchhoff. These theories nevertheless, are deficient as they do not take into account the influence of transverse shear forces on the deformation of plates. Consequently, this work explicitly established a refined plate equation, which considers the effect of shear deformation in line with Vasil'ev approach. The differential equation of plates with shear effect was solved for all edge simply supported (SSSS) condition subjected to uniformly distributed load using orthogonal polynomial method. Numerical solutions in the present study are compared to Navier's solution based on Kirchhoff's hypotheses which gives an insignificant percentage difference for membrane plates (-00.308% for h = 0.05m) and thin plates (00.260% for h = 0.15m). However, it is also observed that the percentage difference between the present study and Kirchhoff's hypotheses is high for fairly thick plates (04.172% for h = 0.40m) due to incorporation of share deformation in the solution. The result presented shows good approximation for analyses of fairy thick and thick Isotropic rectangular plates.

Keywords: Orthogonal, Polynomial, Isotropic Plates, Shear Deformation.

INTRODUCTION

Plates as structural elements are widely employed in many engineering applications and are subjected to variety of loads, including acoustic excitations. Geometrically, plates are surrounded either by straight or curve boundaries. The static or dynamic loads applied on plates are predominantly perpendicular to the plate faces, which are referred to as transverse loads on the plates. The first acceptable theory of bending plates developed [1], reflected plate thickness in the general plate equation as a function of rigidity, D. In addition, the work presented an exact method, which transformed the differential equation into algebraic expressions by use of Fourier trigonometric series. In 1876, an essential thesis on thin plates was published [2]. In this work, two independent basic assumptions that are now generally accepted in the plate-bending theory and are acknowledged as "Kirchhoff's hypothesis" were stated. It revealed that there are just two boundary conditions on a plate edges.

Kirchhoff's hypothesis allowed for the establishment of the classical bending theory of thin plates, which for several decades has been the basis for the calculation and design of structures in numerous areas of engineering and has produced significant theoretical and numerical results. Nevertheless, just as for any other approximation theory, the theorem has some downsides and deficiencies. The most significant assumption of his plate theory is that normal to the middle surface stay normal to the deflected mid-plane and straight. Since this theory disregards the deformation produced by transverse shear, it would result to significant errors if applied to moderately thick plates. For such plates Kirchhoff's theory under-estimates deflections and over-estimates frequencies and buckling loads.

Several scholars have attempted to refine Kirchhoff's theory and such endeavors continue to this day. The most significant advances in this direction were recorded in the literature [3, 4] whose theories take into account the effect of the transverse shear deformation on the deflection of the plate and results to third-order system of governing differential equations, and single equation in terms of bending deflection as a fundamental variable in the analysis of moderately thick plates. These theories are free from the drawbacks of Kirchhoff's theory. However, they are somewhat similar to Mindlin's theory that used a displacement-based approach and proposed a theory where transverse shear stress is assumed to be constant through the thickness of the plate [5]. Although, this violates the shear stress free surface conditions, it satisfies constitutive relations for transverse shear stresses and shear strains in an approximate manner by way of using shear correction factor.

The approach employed by Shi made use of third-order shear deformation theory for the analysis of shear flexible plates and it satisfied constitutive relations between shear stress and shear strain and his theory was based on a more rigorous kinematics of displacements and requires computational effort to obtain solution for more complex boundary conditions of thin plates [6].

The earlier approach attempted improvement to Kirchhoff's theory [2,7]. It assumes uniform distribution of shear force through the thickness of the plate, and to remedy the effects of the assumption, introduces a numerical factor, which needs to be adjusted. A formulation centred on displacement approach was recorded, which does not necessitate shear correction factor [8]. However, the general equations for the motion of a plate given are same as those obtained by Mindlin's theory, so long as the shear coefficient value related is taken as 5/6. Also [9]'s work estimates static analysis of an isotropic rectangular plate with different edge conditions using direct variation method with respect to Ritz to obtain the total potential energy of plate. The characteristics orthogonal polynomials (COP) were used in formulating the shape function of fourth order for the plate under uniformly distributed load for different edge condition. The results gotten by this approach are very close with the results obtained from the exact solutions of classical method. A refined nonlinear shear deformation of thick rectangular plate was derived using an improved mixed variational formulation. The general and direct variation method was used to obtain the total potential energy of the plate by applying the static elastic theory to get the general and direct governing differential equations with its associated boundary conditions respectively [10].

The present investigation made use of the approach proposed by Vasil'ev for obtaining the governing differential equation of the refined plate bending theory, which considered the effect of transverse shear deformation [11]. According to this method, the equations are derived from the equations of the theory of elasticity and contain physical hypothesis.

The General Differential Equations of the Refined Plate Bending Theory

Consider equilibrium of a plate element dx, dy which is subjected to a vertically distributed load P(x,y) applied to an upper surface of the plate as shown in Figure 1. Since the stress resultants and stress couples are anticipated to be applied to the middle plane of this element, a distributed load P(x,y) is transferred to the mid plane. Note that as the element is very small, the moment and force components may be considered to be distributed uniformly over the mid plane of the plate element.

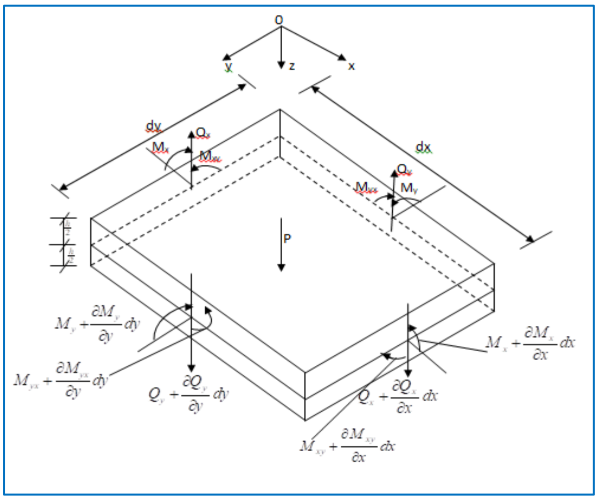

Fig-1: Shear forces and moments acng on the plate element.

Vasil'ev [11] gave the general differential equation of the plate shown above as follows:

$$D\nabla^2\nabla^2\emptyset = P \dots\dots\dots\dots (1)$$

$$\nabla^2\Psi - K^2\Psi = 0 \dots\dots\dots (2)$$

In addition, the deflection of the plate is given as:

$$w = \emptyset - \frac{D}{C}\nabla^2\emptyset \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots (3)$$

Where, P is the transverse load on the plate, $\emptyset$ is the stress function of the plate, Ψ is the stream function.

$$K^2 = \frac{2C}{D(1-\mu)} \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots (4)$$

Where D represents the flexural rigidity of the plate expressed in Eq. (5). C refers to the shear stiffness of the plate in the planes xz and yz. Bearing in mind the equilibrium of the plate element as shown in Figure 1, it is expressed as in Eq. (6):

$$D = \frac{Eh^3}{12(1-\mu^2)} \dots\dots\dots\dots\dots\dots\dots\dots\dots (5)$$

$$C = Gh \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots (6)$$

Where,

$$G = \frac{E}{2(1+\mu)} \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots (7)$$

Thus,

$$C = \frac{hE}{2(1+\mu)} \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots (8)$$

Where E, μ and h are the Young's modulus of elasticity, Poisson's ratio and thickness of the plate respectively.

Characteristic Orthogonal Polynomials in Plates

The stress function for rectangular plate, which is a two-dimensional structure in x and y-axes is considered to be a product of two independent functions; x and y such that:

$$\emptyset(x,y) = F(x) \,.\, G(y)$$

$$= \sum_{m=0}^{\infty} . \sum_{n=0}^{\infty} X_m x^m Y_n y^n \dots\dots\dots\dots\dots\dots (9)$$

Expressing this equation in the form of non-dimensional parameters, say R and Q for x and y directions respectively.

Let;

$$x = aR;\ y = bQ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots (10)$$

Where a, is the dimension of a rectangular plate along x-axis and b, is along y-axis.

Then,

$$\emptyset(R,Q) = \sum_{m=0}^{\infty} . \sum_{n=0}^{\infty} A_m B_n R^m Q^n \dots\dots\dots\dots\dots (11)$$

Where,

$$A_m = X_m a^m,\ B_n = Y_n b^n \dots\dots\dots\dots\dots\dots\dots\dots\dots (12)$$

Since the deflection equation of plate, is a fourth order differential and the density of the plate is constant, then, the value of m and n in Eq. (11) must be equal to 4.If the variation of loading is linear or a second-degree parabola, the value of the power m and n will then be 5or 6 respectively [12].

Expanding Eq. (11) yields:

$$\emptyset(R,Q) = (A_0 + A_1R + A_2R^2 + A_3R^3 + A_4R^4)(B_0 + B_1Q + B_2Q^2 + B_3Q^3 + B_4Q^4) \dots\dots (13)$$

Eq. (13) represents the general stress function for rectangular plates. Where the coefficients A_m and B_n of the series are determined from the boundary conditions at the edges of the plate Polynomial functions were in works of [13, 14] for analysis of stiffened plate and they obtained good approximate solutions.

Edge Conditions:

The deflections and Moment at all edges are equal zero.

$$\emptyset_R(R=0) = 0, A_0 = 0, \emptyset_Q(Q=0) = 0, B_0 = 0, \dots\dots\dots\dots\dots\dots\dots\dots\dots (14)$$

$$\emptyset_R^{II}(R=0) = 0, A_2 = 0, \emptyset_Q^{II}(Q=0) = 0, B_2 = 0, \dots\dots\dots\dots\dots\dots\dots\dots\dots (15)$$

$$\emptyset_R(R=1) = 0, 0 = (A_1 + A_3 + A_4) \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots (16)$$

$$\emptyset_Q(Q=1) = 0, 0 = (B_1 + B_3 + B_4) \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots (17)$$

$$\emptyset_R^{II}(R=1) = 0, A_3 = -2A_4, \emptyset_Q^{II}(Q=1) = 0, B_3 = -2B_4 \dots\dots\dots\dots\dots (18)$$

Then, solving Eqs (16) - (18) by substitution method independently with respect to their directions, yields;

$$\emptyset(R,Q) = A(R - 2R^3 + R^4)(Q - 2Q^3 + Q^4) \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots (19)$$

$$A = A_4Q_4$$

Application of the Refined Theory by Orthogonal Polynomial Method

Consider the classical frequently encountered problem: a simply supported rectangular plate with sides a and b $(0 \le x \le a; 0 \le y \le b)$ subjected to uniformly distributed load of intensity P as shown in Fig-2:

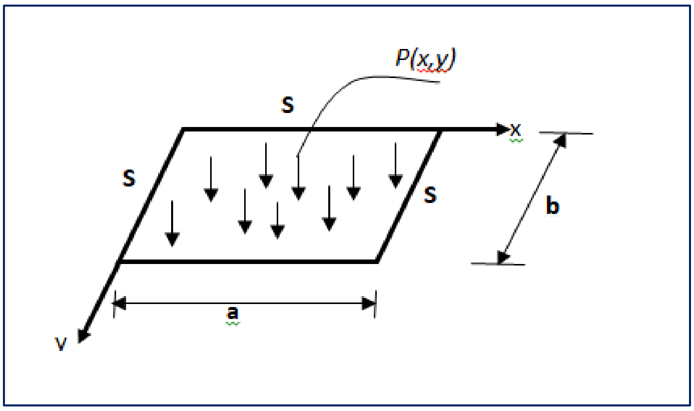

Fig-2: SSSS Rectangular

The stress function as in buckling and post buckling regime using the boundary conditions of SSSS plate was obtained [12] as:

$$\emptyset(R,Q) = A(R-2R^3+R^4)(Q-2Q^3+Q^4) \ldots \ldots (20)$$

Where A is the amplitude of the load, R and Q are non-dimensional terms in x and y directions respectively.

Substituting Eq. (20) into Eq. (1) yields:

$$\left[\frac{d^2}{dx^2}+\frac{d^2}{dy^2}\right]\left[\frac{d^2}{dx^2}+\frac{d^2}{dy^2}\right][(A(R-2R^3+R^4)(Q-2Q^3+Q^4)] = \frac{P}{D} \ldots \ldots (21)$$

In non-dimensional terms, $x = aR, y = bQ, dx = adR, dy = bdQ$.
For $0 \leq R \leq 1; 0 \leq Q \leq 1$
Hence, Eq. (21) can be expressed as:

$$\left[\frac{d^2}{a^2dR^2}+\frac{d^2}{b^2dQ^2}\right]\left[\frac{d^2}{a^2dR^2}+\frac{d^2}{b^2dQ^2}\right][A(R-2R^3+R^4)(Q-2Q^3+Q^4)] = \frac{P}{D} \ldots \ldots (22)$$

Expanding and simplifying Eq. (22) gives:

$$A\left[\frac{24}{a^4}(Q-2Q^3+Q^4)+\frac{2}{a^2b^2}(-12R+12R^2)(-12Q+12Q^2)+\frac{24}{b^4}(R-2R^3+R^4)\right] = \frac{P}{D} \ldots \ldots (23)$$

The aspect ratio

$$r = \frac{a}{b} \ldots \ldots (24)$$

Hence,

$$a = rb \ldots \ldots (25)$$

Substituting Eq. (25) into Equation (23) yields:

$$A\left[\frac{24}{r^4b^4}(Q-2Q^3+Q^4)+\frac{2}{r^2b^4}(-12R+12R^2)(-12Q+12Q^2)+\frac{24}{b^4}(R-2R^3+R^4)\right] = \frac{P}{D} \ldots \ldots (26)$$

Simplifying Eq. (26) gives:

$$A = \frac{Pb^4}{D\left[\frac{24}{r^4}(Q-2Q^3+Q^4)+\frac{2}{r^2}(-12R+12R^2)(-12Q+12Q^2)+24(R-2R^3+R^4)\right]} \ldots \ldots (27)$$

Also substituting Eq. (20) into Eq. (3) yields:

$$w = A(R-2R^3+R^4)(Q-2Q^3+Q^4) - \frac{D}{C}\left[\frac{d^2}{a^2dR^2}+\frac{d^2}{b^2dQ^2}\right][A(R-2R^3+R^4)(Q-2Q^3+Q^4)] \ldots \ldots (28)$$

Simplifying Eq. (28) gives:

$$w = A(R-2R^3+R^4)(Q-2Q^3+Q^4) - \frac{D}{C}\left[\frac{1}{a^2}.A(-12R+12R^2)(Q-2Q^3+Q^4)+\frac{1}{b^2}.A(R-2R^3+12R^4)(-12Q+Q^2)\right] \ldots \ldots (29)$$

Substituting Eq. (25) into Eq. (29) yields:

$$w = A\left[(R-2R^3+R^4)(Q-2Q^3+Q^4) - \frac{D}{b^2C}\left[\frac{1}{r^2}(-12R+12R^2)(Q-2Q^3+Q^4)+(R-2R^3+12R^4)(-12Q+Q^2)\right]\right] \ldots \ldots (30)$$

Substituting Eq. (27) into Eq. (30) gives:

$$w = \left[\frac{Pb^4}{D\left[\frac{24}{r^4}(Q-2Q^3+Q^4)+\frac{2}{r^2}(-12R+12R^2)(-12Q+12Q^2)+24(R-2R^3+R^4)\right]}\right]\left[(R-2R^3+R^4)(Q-2Q^3+Q^4) - \frac{D}{b^2C}\left[\frac{1}{r^2}(-12R+12R^2)(Q-2Q^3+Q^4)+(R-2R^3+12R^4)(-12Q+Q^2)\right]\right] \ldots \ldots (31)$$

Eq. (31) represents the equation of the deflected surface of a SSSS rectangular plate under transverse shear.

At maximum deflection,

$$R = Q = 0.5 \ldots\ldots\ldots\ldots\ldots\ldots (32)$$

Substituting the above values for R and Q in Eq. (32) into Eq. (31) yields:

$$w_{max} = \left[\frac{Pb^4}{D\left(\frac{7.5}{r^4} + \frac{18}{r^2} + 7.5\right)}\right]\left[0.0977 - \frac{D}{b^2C}\left(-\frac{0.9375}{r^2} - 0.9375\right)\right] \ldots\ldots.\ldots.33)$$

For a square plate, i.e. $a = b$, the aspect ratio, $r = 1$.

Hence;

$$w_{max} = \frac{Pb^4}{33D}\left[0.0977 + \frac{1.875D}{b^2C}\right] \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (34)$$

Where b is the span of the plate
Substituting Eq. (5) and Eq. (8) into Eq. (34) and simplifying, yields:

$$w_{max} = \frac{12Pb^4(1-\mu^2)}{33Eh^3}\left[0.0977 + \frac{1.875h^2(1+\mu)}{6b^2(1-\mu^2)}\right] \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots.\ldots (35)$$

The maximum deflection using orthogonal polynomial is affected only by the first four series of the shape function of the deflected surface w(x,y); hence a numeric constant (7/5) is used to multiply the value of w_{max} in other to reduce the small deviation from the exact solution.

However, according to Navier, w_{max} for simply supported plates based on Kirchoff's hypothesis without the effect of transverse shear deformation given by Timoshenko and Woinowsky – Krieger [15] can be written as;

$$w_{max} = \frac{12P}{\pi^6 D\left(\frac{1}{a^2} + \frac{1}{b^2}\right)^2} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (36)$$

Eq. (36) can be expressed as;

$$w_{max} = \frac{192P(1-\mu^2)}{\pi^6 Eh^3\left(\frac{1}{a^2} + \frac{1}{b^2}\right)^2} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (37)$$

Where a and b are length and width of the plate respectively.

RESULTS AND DISCUSSION

The following physical and geometric properties were adopted: a = b = 4m, $\mu = 0.3$, $P = 200KN$, $E = 205MPa$, with arange of plate's thickness of $0.05 \le h \le 1.00$, h(m). The maximum deflection is evaluated based on the above data and the result of the present study compared to Navier's [1, 15] solution based on Kirchhoff's hypothesis as shown in Table 1 below. A close observation of these numerical solutions in Table 1 shows clearly the disparity between the present study and Kirchhoff's hypotheses. It is observed that as the plate's thickness increases, the solutions for the deflection obtained using the refined theory takes cognizance of shear deformation, which agrees with the technical literature for elastic plates. It is also observed that there is a close agreement in the numerical solutions between Kirchhoff's plate theory and the plate theory presented that accounts for shear deformation for membrane plates and thin plates which resulted to an insignificant percentage difference of -00.308% for h = 0.05m and 00.260% for h = 0.15m. This also agrees with the technical literature for elastic plate. However, there is a significant percentage difference between Kirchhoff's hypotheses and the present study in fairly thick plates (04.172% for h = 0.40m). This is as a result of the negligence of transverse shear acting at the edges of the plate by Kirchhoff's hypotheses.

Table-1: Variable deformation of elastic all round simply supported (SSSS) plate

S/N	Plate's Thickness, h (m)	Navier's solution based on Kirchoff's assumption, w (m)	Present study, w (m)	Percentage difference (%)
1	0.05	9.0780E 01	9.0500E 01	-00.308
2	0.10	1.1347E 01	1.1347E 01	-00.095
3	0.15	3.3622E 00	3.3709E 00	00.260
4	0.20	1.4184E 00	1.4292E 00	00.758
5	0.25	7.2624E-01	7.3640E-01	01.398
6	0.30	4.2028E-01	4.2944E-01	02.180
7	0.35	2.6466E-01	2.7288E-01	03.105
8	0.40	1.7730E-01	1.8470E-01	04.172
9	0.45	1.2452E-01	1.3123E-01	05.381
10	0.50	9.0780E-02	9.6892E-02	06.732
11	0.55	6.8204E-02	7.3815E-02	08.226
12	0.60	5.2532E-02	5.7716E-02	0.9.862
13	0.65	4.1320E-02	4.6130E-02	11.640
14	0.70	3.3083E-02	3.7569E-02	13.560
15	0.75	2.6898E-02	3.1100E-02	15.623
16	0.80	2.2163E-02	2.6114E-02	17.828
17	0.85	1.8477E-02	2.2205E-02	20.175
18	0.90	1.5566E-02	1.9094E-02	22.664
19	0.95	1.3235E-02	1.6583E-02	25.296
20	1.00	1.1347E-02	1.4533E-02	28.070

CONCLUSION

The results of the analysis above show clearly that:

a) The solution that made use of orthogonal polynomial function presented is adequate in analyzing both fairly thick and thick isotropic rectangular plates.

b) Transverse shear force has a significant influence on the deformation of elastic plates and should not be ignored.

c) The equation for the deformation of plates of Kirchhoff and Navier [1, 2] lacks the shear stiffness term that predicts the influence of shear deformation on elastic plates, hence, does not give accurate results for the deformation of fairly thick elastic plates.

d) The solution for transverse shear force on the deformation of SSSS plate under uniformly distributed load presented using orthogonal polynomial function can be extended for plates with other boundary conditions

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