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On The Homogeneous Biquadratic Equation with Four Unknowns $x^4+y^4+z^4=32w^4$

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Abstract: The bi-quadratic equation with 4 unknowns given by $x^4+y^4+z^4=32w^4$ is analyzed for its patterns of non-zero distinct integral solutions. Six different patterns of integer solutions to the above bi-quadratic equation are presented. A few interesting relations between the solutions and special numbers, namely, polygonal number and pyramidal number are exhibited.
Keywords: Homogeneous bi-quadratic, bi-quadratic equation with four unknowns, integer solutions polygonal and pyramidal numbers.
MSC Subject classification: 11D41.

INTRODUCTION

There is a great interest for Mathematicians since ambiguity in homogeneous and non-homogeneous Bi-quadratic Diophantine Equations [1-3]. In this context, one may refer [4-18] for varieties of problems on the biquadratic Diophantine equations with three and four variables. In this paper, biquadratic equations with four variables given by $x^4+y^4+z^4=32w^4$ is analyzed for its non-zero distinct integer solutions. A few interesting relations between the solutions and special polygonal numbers, pyramidal numbers are exhibited.

NOTATIONS

p_n^m :Pyramidal number of rank n with size m

cp_n^m :Centered pyramidal number of rank n with size m

So_n :Stella Octangular number of rank n

G_n :Gnomonic number of rank n

PR_n :Pronic number of rank n

METHOD OF ANALYSIS

The diophantine equation representing the biquadratic equation with four unknowns under consideration is given by

$$x^4+y^4+z^4=32w^4 \tag{1}$$

Note that (1) is satisfied by the following non-zero integer quadruples
(0,2,2,1),(6,0,6,3),(16,-6,10,7),(30,-16,14,13),(2,0,2,1),(16,-10,6,7),(42,-32,10,19),(80,-66,14,37)

However, we have other solutions for (1), which are illustrated below.
Introducing the transformations

$$x=u+v;\ y=u-v;\ z=2u \tag{2}$$

in (1), it simplifies to

$$v^2+3u^2=4w^2 \tag{3}$$

The above equations (3) is solved through different methods and thus, one obtains distinct patterns of integer solutions of (1)

PATTERN 1

Assume $w = w(a,b) = a^2 + 3b^2$ (4)

where a and b are non-zero distinct integers

Write 4 as $4 = (1+i\sqrt{3})(1-i\sqrt{3})$ (5)

Substituting (4) & (5) in (3) and using the method of factorization, define

$$v + i\sqrt{3}u = (1+i\sqrt{3})(a+i\sqrt{3}b)^2$$

Equating real and imaginary parts, we have

$$v = a^2 - 3b^2 - 6ab$$

$$u = a^2 - 3b^2 + 2ab$$

Hence in view of (2) and (4), the non-zero distinct integral solutions of (1) are given by

$$x = x(a,b) = 2a^2 - 6b^2 - 4ab$$

$$y = y(a,b) = 8ab$$

$$z = z(a,b) = 2a^2 - 6b^2 + 4ab$$

$$w = w(a,b) = a^2 + 3b^2$$

A few interesting properties are presented below:

*) $x(a,b) - z(a,b) + y(a,b) = 0$

*) $2w(a,1) + z(a,1) = 4PR_a$

*) $z(a^2,a^2) + y(a^2,a^2) - x(a^2,a^2)$ is a bi-quadratic integer

*) $z(a,a^2+1) - x(a,a^2+1) = 16CP_a^3$

*) $y(a,a) + z(a,a) + w(a,a) \equiv 0 \pmod{12}$

*) $2[y(a,a) + w(a,a)]$ is a Nasty number

PATTERN 2

The ternary quadratic equation (3) can be written as

$$3u^2 = 4w^2 - v^2 \qquad (6)$$

Factorizing (6), we get

$$(3u)(u) = (2w+v)(2w-v)$$

which is expressed in the form of ratio as

$$\frac{3u}{2w+v} = \frac{2w-v}{u} = \frac{\alpha}{\beta}, \beta \neq 0,$$

This is equivalent to the following two equations.

$$3\beta u - \alpha v - 2\alpha w = 0$$

$$-\alpha u - \beta v + 2\beta w = 0$$

Applying the method of cross multiplication, we get

$$u = -4\alpha\beta$$
$$v = 2\alpha^2 - 6\beta^2$$
$$w = -\alpha^2 - 3\beta^2$$

In view of (2), the corresponding non-zero distinct integer solutions of (1) are

$$x = x(\alpha,\beta) = 2\alpha^2 - 4\alpha\beta - 6\beta^2$$
$$y = y(\alpha,\beta) = -2\alpha^2 - 4\alpha\beta + 6\beta^2$$
$$z = z(\alpha,\beta) = -8\alpha\beta$$
$$w = w(\alpha,\beta) = -\alpha^2 - 3\beta^2$$

Some interesting properties are as follows:

*) $2w(3\beta,\beta) - y(3\beta,\beta) = 0$

*) $2w(a,a+1) - y(a,a+1) \equiv 0 \pmod 4$

*) $x(a,a+1) + y(a,a+1) - z(a,a+1) = 0$

*) $x(a^2,a+1) + y(a^2,a+1) = -16P_a^5$

*) $3[y(a,a) - z(a,a)]$ is a Nasty number

PATTERN-3:

Rewrite (3) as, $4w^2 - 3u^2 = v^2 * 1$ (7)

Write 1 as $1 = (2+\sqrt{3})(2-\sqrt{3})$ (8)

Assume $v = 4a^2 - 3b^2$ (9)

Substituting (8) and (9) in (7) and using the method of factorization, we get

$$(2w + \sqrt{3}u) = (2a + \sqrt{3}b)^2(2+\sqrt{3})$$

Equating rational and irrational parts, we get

$$w = 4a^2 + 3b^2 + 6ab$$
$$u = 4a^2 + 3b^2 + 8ab \qquad (10)$$

Again substituting (9) and (10) in (2), we get

$$x = x(a,b) = 8a^2 + 8ab$$
$$y = y(a,b) = 8ab + 6b^2$$
$$z = z(a,b) = 8a^2 + 6b^2 + 16ab$$
$$w = w(a,b) = 4a^2 + 3b^2 + 6ab$$

A few interesting properties are presented below:

*) $z(a(a+1),(a+2)) - 2w(a(a+1),(a+2)) = 24p_a^3$

*) $13[w(a,a) - z(a,a) + x(a,a) + y(a,a)]$ is a perfect square

*) $z(a^2,a+1) + y(a^2,a+1) \equiv 0 \pmod 4$

*) $z(a^2,a+1) - 2w(a^2,a+1) = 8P_a^5$

*) $y(a,1) + z(a,1) - w(a,1) = perfectsquare + 9Gn_a$

*) $3[x(a,a) - y(a,a)]$ is a Nasty number

PATTERN-4:

Introducing the linear transformations $u = X + T, v = X - 3T$ (11)

in (3), it leads to $X^2 + 3T^2 = w^2$ (12)

which is satisfied by

$$X = 3p^2 - q^2$$

$$T = 2pq$$

$$w = 3p^2 + q^2$$

Substituting the above values in (11), we get

$$u = u(p,q) = 3p^2 - q^2 + 2pq$$

$$v = v(p,q) = 3p^2 - q^2 - 6pq$$

In view of (2), we have

$$x = x(p,q) = 6p^2 - 4pq - 2q^2$$

$$y = y(p,q) = 8pq$$

$$z = z(p,q) = 6p^2 + 4pq - 2q^2$$

$$w = w(p,q) = 3p^2 + q^2$$

A few interesting properties are presented below:

*) $z(a,2a^2 - 1) - x(a,2a^2 - 1) = 8SO_a$

*) $y(a,a) - (a,a)$ is a perfect square

*) $z(a^2, a+1) + y(a^2, a+1) - x(a^2, a+1) = 32p_a^5$

*) $[108y(a^2,a^2) + w(a^2,a^2)]$ is a bi quadratic itneger

*) $x(a,a) - w(a,a) + z(a,a)$ is a Nasty number

PATTERN-5:

Rewrite (3) as $v^2 = 4w^2 - 3u^2$ (13)

Introducing the linear transformations $u = X + 4T, w = X + 3T$ (14)

Substituting (14) in (13), we get $X^2 = 12T^2 + v^2$ (15)

Following procedure presented in patteren-4, the corresponding integer values are as follows

$$x = x(p,q) = 24p^2 + 8pq$$

$$y = y(p,q) = 8pq + 2q^2$$

$$z = z(p,q) = 24p^2 + 16pq + 2q^2$$

$$w = w(p,q) = 12p^2 + q^2 + 6pq$$

PATTERN-6

It is worth to note that (15) can also be written as the system of double equations as follows:

$System-1$	$System-2$	$System-3$	$System-4$
$X+v=T^2$	$X+v=2T^2$	$X+v=6T^2$	$X+v=3T$
$X-v=12$	$X-v=6$	$X-v=2$	$X-v=4T$

Solving each of the above systems, the values of X,T,v are obtained. Using these values respectively in (14) and employing (2), one obtains the corresponding integer solutions to (1).

CONCLUSION

In this paper, we have presented infinitely many non-zero integral solutions for the non-homogeneous biquadratic equation with four unknowns $x^4+y^4+z^4=32w^4$. As biquadratic equations are rich in variety, one may consider the other forms of biquadratic equations with variables greater than or equal to 4 and search for their corresponding integer solutions.

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